04/05/16

Introduction to the Langlands Program

$G = Gl_n$

$n = 1$, Kronecker ($\cancel{F = \mathbb{Q}}$): F number field

~~(both sides are trivial!)~~

$$\left\{ \begin{array}{l} \text{Unramified} \\ \text{characters} \\ \text{of } \cancel{Gal(\overline{\mathbb{Q}}/\mathbb{Q})} \\ \quad Gal(\overline{F}/F) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Characters of} \\ \underset{F^*}{\cancel{\mathbb{Q}^*}} \backslash Gl_1(\underset{F}{\mathbb{A}}) / \prod_p Gl_1(\underset{\mathcal{O}_v}{\cancel{\mathbb{Z}_p}})\, \underset{\cdots}{\cancel{\mathbb{R}^*_{>0}}} \end{array} \right\}$$

(abelian class field theory)

$$\left\{ \begin{array}{l} \text{characters} \\ \text{of } Gal(\overline{\mathbb{Q}}/\mathbb{Q} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Characters of} \\ \mathbb{Q}^* \backslash Gl_1(\mathbb{A}) / K\, \mathbb{R}^*_{>0} \end{array} \right\}$$

with $K \hookrightarrow \prod_p Gl_1(\mathbb{Z}_p)$ finite index subgp

$$\begin{tikzcd} \text{characters of } Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[d, phantom, "\Downarrow"] & \text{Characters of } \ldots \arrow[d, phantom, "\Downarrow"] \\ Frob_\ell \arrow[r, leftrightarrow] & (1, \ldots, 1, \underset{\uparrow\, \ell\text{-th place}}{\pi_\ell}, 1, \ldots, 1) \end{tikzcd}$$

Frobenius:

$$\begin{tikzcd} F \arrow[r, hookleftarrow] \arrow[d, dash] & \mathcal{O}_F \arrow[r, two heads] \arrow[d, dash] & \mathcal{O}_F/\lambda = \mathbb{F}_{\ell^r} \arrow[d, dash, "r"] \\ \mathbb{Q} \arrow[r, hookleftarrow] & \mathbb{Z} \arrow[r, two heads] & \mathbb{Z}/\ell = \mathbb{F}_\ell \end{tikzcd}$$

$$\overline{Frob_\ell^{-1}}(x) = \bar{x}^\ell \quad \text{for } x \in \mathcal{O}_F.$$

<u>NB</u>. $Frob_\ell \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ only unique up to conjugation!

$n > 1$, Langlands:

$$\left\{ \begin{array}{l} \text{unramified representations } E \\ \text{of } Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \text{ of rank } n \end{array} \right\} \begin{array}{c} \longrightarrow \\ \dashleftarrow \end{array} \left\{ \begin{array}{l} \text{representations of } G(\mathbb{A}) \\ \text{on } L^2(G(\mathbb{Q})\backslash G(\mathbb{A})) \text{ with a} \\ \text{nonzero vector fixed by } K = \prod_p G(\mathbb{Z}_p) \\ \text{+ some technical conditions} \\ \qquad \text{on center} \end{array} \right\}$$

$$n \text{ eigenvalues of } Frob_\ell \longleftrightarrow \text{Satake parameters } \alpha_1, \ldots, \alpha_n \text{ at } \ell$$

Classical picture, $n = 2$; $F = \mathbb{Q}$:

holomorphic modular forms $f : \mathbb{H}^{\pm} = \mathbb{C} \setminus \mathbb{R} \longrightarrow \mathbb{C}$

$$f\left(\frac{az+b}{cz+d}\right) = \overbrace{(cz+d)}^{=: \mathcal{J}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z\right)}{}^{k} \cdot f(z)$$

$$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Gl(2, \mathbb{Z})$$

$$f(z+1) = f(z) \rightsquigarrow f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

$\uparrow$ cuspidal $\Rightarrow L^2$

Hecke operators for ℓ prime: $T_\ell f = \lambda_\ell f$

$\xRightarrow[\text{Hecke}]{}$ a_n is a number theoretic fct^n,

ie $a_{mn} = a_m a_n$ for $(m,n) = 1$

Write $1 - a_p p^{-s} + p p^{k-1-2s} = (1 - p^{\frac{k-1}{2}-s} \alpha_1)(1 - p^{\frac{k-1}{2}-s} \alpha_2)$ $\underset{\text{Ramanujan}}{\rightsquigarrow}$ $|\alpha_1| = |\alpha_2| = 1$

$a_{\ell^d} = \ell^{\text{power?}} (\alpha_1^d + \alpha_1^{d-1} \alpha_2 + \dots + \alpha_2^d)$ $\qquad a_\ell \overset{!}{=} \lambda_\ell$

(α_1, α_2 depend on ℓ)

$\wr$

"Satake parameters"

---

f hol. modular of wt k

$g \in G(\mathbb{R}) \rightsquigarrow h(g) := \mathcal{J}(g, i)^{-k} \cdot f(g(i))$ gives $h : G(\mathbb{R}) \longrightarrow \mathbb{C}$

$$L^2(Gl_2(\mathbb{Z}) \backslash Gl_2(\mathbb{R})) = L^2_{\text{discrete}} \oplus L^2_{\text{cont}}$$

and h defines an irreducible rep^n in L^2_{discrete}.

Now we $Gl_n(\mathbb{Q}) \backslash Gl_n(\mathbb{A}) \simeq Gl_n(\mathbb{Z}) \backslash Gl_n(\mathbb{R}) \times \prod_p Gl_n(\mathbb{Z}_p)$ $\quad \dots$

$$L_f(s) := L_\infty(s)\cdot \prod_{\ell} \frac{1}{1-\tilde{a}_p p^{-s}+p^{-s}} \qquad \tilde{a}_p = \frac{a_p}{p^{\frac{k-1}{2}}} = \alpha_1+\alpha_2$$

$$= L_\infty(s)\cdot \sum_{n=1}^{\infty} \tilde{a}_n n^{-s}$$

Functional equation:

$$L_f(1-s) = \text{"easy constant"} \cdot L_{\bar{f}}(s)$$

Hecke's converse thm: $\sum_{n=1}^{\infty} \tilde{a}_n n^{-s}$ for given $\tilde{a}_n \in \mathbb{C}$, convergent for $Re(s) > 1$, with holomorphic continuation of $(L_\infty(s) \cdot \sum_{n=1}^{\infty} \tilde{a}_n n^{-s})$ to $\mathbb{C}$ and the correct functional eq^n & growth properties

$\Rightarrow \sum_{n=1}^{\infty} \tilde{a}_n n^{-s}$ comes from a modular form.

- Hecke converse thm
- Grothendieck étale Lefschetz formula
- trace formula

$\Big\} \Rightarrow$ Langlands reciprocity for all n over function fields F (finite extensions of $\mathbb{F}_p(t)$)

$\mathbb{Q} \rightsquigarrow \mathbb{F}_p(t) = fct^n$ field of $\mathbb{P}^1 / \mathbb{F}_q$

$F \rightsquigarrow fct^n$ field of smooth proj curve $^X/\mathbb{F}_q$

Thm ($n=2$ Drinfeld, $n \geq 3$ (general) Lafforgue '02, $n \geq 3$ (unramified) Frenkel-Gaitsgory-Vilonen '03) Let $\Lambda = \overline{\mathbb{Q}_\ell}$

$$\left\{ \begin{array}{c} n\text{-dim irred. unramified} \\ \Lambda\text{-adic representations } E \\ \text{of } Gal(\bar{F}/F) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{irreducible cuspidal unramified} \\ \text{automorphic representations} \\ \text{on } L^2(G(\mathbb{F})\backslash G(\mathbb{A}_F)) \end{array} \right\}$$

+ condition on $\updownarrow$ determinant $\qquad$ + condition on centre

$(\pi_1^{et}(X, x_0) \longrightarrow Gl_\Lambda(E))$

$\mathcal{E} \rightsquigarrow$ Λ-adic sheaf on the curve X (a local system) $\rightsquigarrow$ L-fct^n $L_{\mathcal{E}}(s)$ rational fct^n in p^{-s} with functional eq^n + good analytic properties

eg $X = \mathbb{P}^1$, $\mathcal{E} = \Lambda_X$ $\rightsquigarrow$

$$L(s) = \frac{1}{\underset{\substack{\wr \\ H^0(X,\Lambda)}}{(1-p^{-s})}\underset{\substack{\wr \\ H^2(X,\Lambda)}}{(1-pp^{-s})}}$$

$H^1(X,\Lambda) = 0$

This gives "$\longrightarrow$" via converse thm.

"$\longleftarrow$" much more difficult, via trace formula. (See Lafforgue)

Geometric Langlands

Replace $X/\mathbb{F}_q$ by a curve X/k with k arbitrary field.

How to make the passage ?

Dictionary:

- $Gl_n(F)\backslash Gl_n(\mathbb{A}_F)/\prod_v Gl_n(\mathcal{O}_v)$ $\rightsquigarrow$ Bun_n (stack of $\mathcal{O}_X$-v.bundles of rank n)

  $\left[n=1: \quad {}_{F^*}\backslash^{\mathrm{Div}(X)} \rightsquigarrow \mathrm{Pic}(X) \right]$

- fct^{ns} f on $\ldots\backslash^{\cdots}/\ldots$ $\rightsquigarrow$ complexes of sheaves (Λ-adic) $\mathcal{F}$ on Bun_n

  $\left[f^{\mathcal{F}}(x) = \mathrm{str}(\mathrm{Frob}_x, \mathcal{F}_x) \longleftarrow \mathcal{F} \right]$

- L^2-condition $\rightsquigarrow$ perversity condition $\mathcal{F} \in \mathrm{Perv}(\mathrm{Bun}_n)$

- Hecke operators $\rightsquigarrow$ correspondences

$$\begin{tikzcd} & \mathrm{Hecke}_i \arrow[dl, "u"'] \arrow[dr, "v"] & \\ \mathrm{Bun}_n & & X \times \mathrm{Bun}_n \end{tikzcd}$$

Here $\mathcal{H}ecke_i$: parametrizes $(\mathcal{M}, \mathcal{M}', x, \beta)$

where $\mathcal{M}, \mathcal{M}'$ are locally free $\mathcal{O}_X$-modules of rk n

and $\beta : \mathcal{M}' \hookrightarrow \mathcal{M} \twoheadrightarrow$ skyscraper isomorphic to $(\mathcal{O}_X(\mathcal{O}_X(-x))^{\oplus i}$

$\rightsquigarrow H_i : \mathcal{D}^b_c(\mathrm{Bun}_n) \longrightarrow \mathcal{D}^b_c(X \times \mathrm{Bun}_n)$.

$\shortparallel$

Thus $\quad R v_* \, u^* [\mathrm{reldim}(u)]$

$$\left\{ \begin{array}{l} E \text{ representation of } \pi_1(X, x_0) \\ \text{of dimension } n \text{ over } \Lambda = \overline{\mathbb{Q}}_\ell \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{perverse } \underline{\text{eigensheaf}} \ \mathrm{Aut}_E \\ \text{on } \mathrm{Bun}_n, \text{ ie} \\ H_i(\mathrm{Aut}_E) \cong \Lambda^i(E) \boxtimes \mathrm{Aut}_E \end{array} \right\}$$